

# Introduction to QPA

## Part 1

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Third GAP Days

# Outline

- 1 About QPA
  - What is QPA?
  - Obtaining QPA
- 2 Basic structures
  - Quivers
  - Path algebras
  - Modules

# What is QPA?

- QPA = Quivers and Path Algebras
- GAP package for computations with quotients of path algebras and their modules

# Obtaining QPA

- QPA is distributed with GAP (from version 4.7.8)
- Can also clone git repository from  
`https://github.com/gap-system/qpq`  
to follow QPA development
- Loading QPA in GAP:  
`gap> LoadPackage("qpq");`

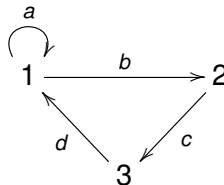
# Basic structures

- Quivers
- Path algebras (modulo relations)
- Modules (representations)  
and homomorphisms

# Quivers

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

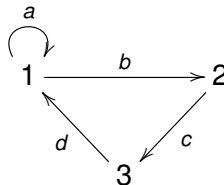
$$1 \begin{matrix} \xrightarrow{a} \\ \xrightarrow{b} \end{matrix} 2$$



# Quivers

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$1 \begin{matrix} \xrightarrow{a} \\ \xrightarrow{b} \end{matrix} 2$$



*Quiver*: oriented graph (loops and multiple edges allowed)

# Paths

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Paths in  $Q$ :



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Paths in  $Q$ :

- Length 0:  $e_1, e_2, e_3$  (vertices/trivial paths)
- Length 1:  $a, b$  (arrows)
- Length 2:  $ab$  (concatenation of  $a$  and  $b$ )

# Paths

$$Q: 1 \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} 2$$

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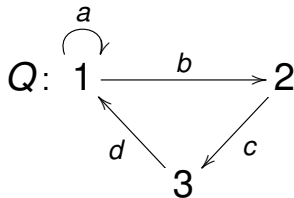
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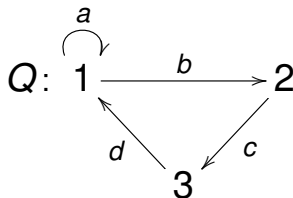
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# Paths



Paths in  $Q$ :

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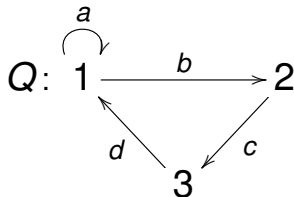


Paths in  $Q$ :

- Length 0:  $e_1, e_2, e_3$



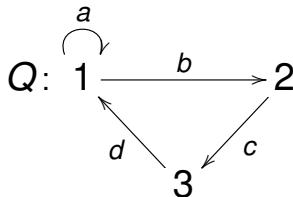
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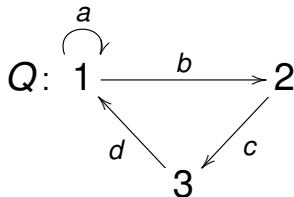
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Paths in  $Q$ :

- Length 0:  $e_1, e_2, e_3$
- Length 1:  $a, b, c, d$
- Length 2:  $a^2, ab, bc, cd, da, db$

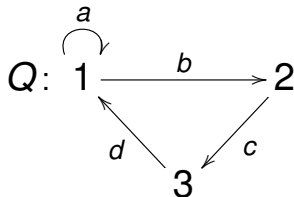
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- Length 3:  $a^3, a^2b, abc, bcd, cda, cdb, da^2, dab, dbc$

# Paths



Paths in  $Q$ :

- Length 0:  $e_1, e_2, e_3$
- Length 1:  $a, b, c, d$
- Length 2:  $a^2, ab, bc, cd, da, db$
- Length 3:  $a^3, a^2b, abc, bcd, cda, cdb, da^2, dab, dbc$
- ...

# Constructing a quiver in QPA

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);  
<quiver with 3 vertices and 2 arrows>
```

# Path algebras

- Given quiver  $Q$  and field  $k$
- Define *path algebra*  $kQ$
- Basis: paths in  $Q$
- Multiplication: concatenation of paths

# Path algebras

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$k$  a field

# Path algebras

$$\left. \begin{array}{l} Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \\ k \text{ a field} \end{array} \right\} \rightsquigarrow \text{path algebra } kQ$$



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- Basis:  $\{e_1, e_2, e_3, a, b, ab\}$
- Multiplication:

$$e_1 \cdot e_1 = e_1$$

$$e_1 \cdot e_2 = 0$$

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$$e_2 \cdot a = 0$$

$$a \cdot b = ab$$

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$$a \cdot b = ab$$

$\cdot$	$e_1$	$e_2$	$e_3$	$a$	$b$	$ab$
$e_1$	$e_1$	0	0	$a$	0	$ab$
$e_2$	0	$e_2$	0	0	$b$	0
$e_3$	0	0	$e_3$	0	0	0
$a$	0	$a$	0	0	$ab$	0
$b$	0	0	$b$	0	0	0
$ab$	0	0	$ab$	0	0	0

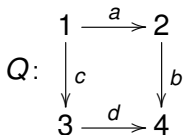
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gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);  
<quiver with 3 vertices and 2 arrows>  
gap> kQ := PathAlgebra(Rationals, Q);  
<Rationals[<quiver with 3 vertices and 2 arrows>]>
```

# Relations

- *Relation*: linear combination of paths with common source and target

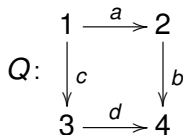


$$\sigma = \underbrace{ab - 2cd} \in kQ$$

source: 1  
target: 4

# Quotient of path algebra modulo relations

- Given path algebra  $kQ$  and a set  $\rho \subseteq kQ$  of relations.
- Can create quotient algebra  $kQ/\langle \rho \rangle$ .



$$A = kQ/\langle ab - 2cd \rangle$$

# Quotients in QPA

$$Q: \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ \downarrow c & & \downarrow b \\ 3 & \xrightarrow{d} & 4 \end{array} \quad A = kQ / \langle ab - 2cd \rangle$$

```
gap> Q := Quiver(4, [[1,2,"a"],[2,4,"b"],  
                    [1,3,"c"],[3,4,"d"]]);;  
gap> kQ := PathAlgebra(Rationals,Q);;  
gap> A := kQ/[kQ.a*kQ.b - 2*kQ.c*kQ.d];;  
gap> A.a * A.b;  
[(1)*a*b]  
gap> A.c * A.d;  
[(1/2)*a*b]
```

# Admissible ideals

- $J \subseteq kQ$  ideal generated by the arrows
- Ideal  $I \subseteq kQ$  *admissible* if  $J^t \subseteq I \subseteq J^2$  for some  $t \geq 2$
- If  $I$  admissible, then  $kQ/I$  finite-dimensional



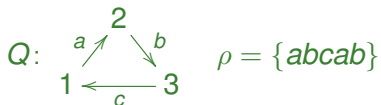
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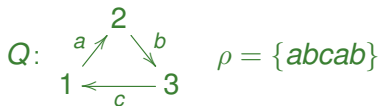


# Admissible ideals – what does it mean?

$$\underline{J^t} \subseteq \underline{I} \subseteq \underline{J^2}$$

long paths must die

no single arrows in relations



# Admissible ideals – why?

Algebras of the form  $kQ/I$  with  $I$  admissible ...

- ... are “almost all” finite-dimensional algebras, and
- ... have a nice theory.

# Checking ideals for admissibility in QPA

```
gap> Q := Quiver(4, [[1,2,"a"],[2,4,"b"],  
                    [1,3,"c"],[3,4,"d"]]);;  
gap> kQ := PathAlgebra(Rationals,Q);;  
gap> I := Ideal(kQ, [kQ.a*kQ.b - 2*kQ.c*kQ.d]);;  
gap> IsAdmissibleIdeal(I);  
true
```

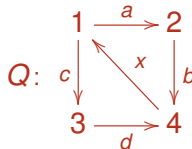
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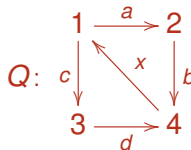
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Take ...

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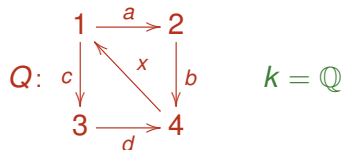
$$k = \mathbb{Q}$$



# Algebras: summary

Take ...

- a QUIVER  $Q$ ,
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- a set  $\rho \subseteq kQ$  of RELATIONS

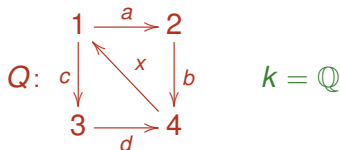


$$\rho = \{ab - cd, dxa\}$$

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Take ...

- a **QUIVER**  $Q$ ,
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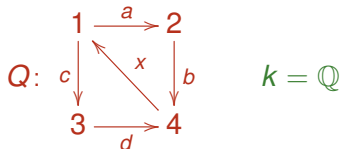
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Let ...  $A = kQ / \langle \rho \rangle$

# Modules and representations

$$\text{mod } kQ \simeq \text{Rep}_k Q$$

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representations of  $Q$  over  $k$



# Representations

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Start with the quiver, and put

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$$R: V_1 \xrightarrow{f_a} V_2 \xrightarrow{f_b} V_3$$

Start with the quiver, and put

- a vector space at each vertex,
- a linear transformation on each arrow.

# A representation

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$R: k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

# Creating a representation in QPA

$$R: k \xrightarrow{(2 \ 0)} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);;  
gap> kQ := PathAlgebra(Rationals, Q);;  
gap> M := RightModuleOverPathAlgebra  
      (kQ, [1,2,1],  
          [["a", [[2,0]]], ["b", [[4],[-1]]]]);  
<[ 1, 2, 1 ]>
```

# Module structure of a representation

$$R: k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

An element  $e$  of  $R$ :

$$e: 3 \longrightarrow \begin{pmatrix} 5 & 7 \end{pmatrix} \longrightarrow 4$$

Multiplying  $e$  with elements of  $kQ$ :

$$e \cdot v_1: 3 \longrightarrow \begin{pmatrix} 0 & 0 \end{pmatrix} \longrightarrow 0$$

$$e \cdot a: 0 \longrightarrow \begin{pmatrix} 6 & 0 \end{pmatrix} \longrightarrow 0$$

$$e \cdot b: 0 \longrightarrow \begin{pmatrix} 0 & 0 \end{pmatrix} \longrightarrow 13$$

# Representations and relations

Given quiver with relations:

$$Q: \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ c \downarrow & & \downarrow b \\ 3 & \xrightarrow{d} & 4 \end{array} \quad \rho = \{ab - 2cd\}$$

A representation  $M$  of  $Q$  respects the relation  $ab - 2cd$  if  $f_a f_b - 2f_c f_d = 0$ .

$$M: \begin{array}{ccc} V_1 & \xrightarrow{f_a} & V_2 \\ f_c \downarrow & & \downarrow f_b \\ V_3 & \xrightarrow{f_d} & V_4 \end{array}$$

# Representations and relations

Representations respecting relation  $ab - 2cd$ ?

$$\begin{array}{ccc}
 k & \xrightarrow{f_a=1} & k \\
 f_c=1 \downarrow & & \downarrow f_b=(1 \ 0) \\
 k & \xrightarrow{f_d=(0 \ 1)} & k^2
 \end{array}$$

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# Representations and relations

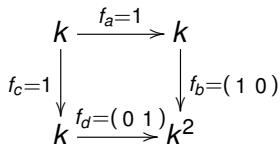
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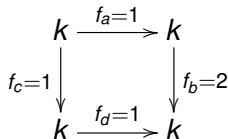
$$\begin{array}{ccc}
 k & \xrightarrow{f_a=1} & k \\
 f_c=1 \downarrow & & \downarrow f_b=2 \\
 k & \xrightarrow{f_d=1} & k
 \end{array}$$

# Representations and relations

Representations respecting relation  $ab - 2cd$ ?



**NO** ( $f_a f_b - 2 f_c f_d = (1 \ -1) \neq 0$ )



**YES** ( $f_a f_b - 2 f_c f_d = 0$ )

# Representations and relations

For  $A = kQ/\langle \rho \rangle$ :

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finitely generated  $A$ -modules

representations of  $Q$  over  $k$   
respecting  $\rho$

# Module homomorphisms

For two modules  $M$  and  $N$  given as representations:

$$M: \quad V_1 \xrightarrow{f_a} V_2 \xrightarrow{f_b} V_3$$

$$N: \quad W_1 \xrightarrow{g_a} W_2 \xrightarrow{g_b} W_3$$

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For two modules  $M$  and  $N$  given as representations:

$$\begin{array}{ccccc} M: & V_1 & \xrightarrow{f_a} & V_2 & \xrightarrow{f_b} & V_3 \\ & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\ N: & W_1 & \xrightarrow{g_a} & W_2 & \xrightarrow{g_b} & W_3 \end{array}$$

... a homomorphism  $h: M \rightarrow N$  is given by:

- linear maps  $h_i$  for every vertex  $i$ ,



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... a homomorphism  $h: M \rightarrow N$  is given by:

- linear maps  $h_i$  for every vertex  $i$ ,
- commuting with the linear maps for the arrows.

# Module homomorphisms in QPA

$$M: \quad 0 \longrightarrow k \xrightarrow{5} k$$

$$N: \quad k \xrightarrow{(0 \ 3)} k^2 \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"], [2,3,"b"]]);;
gap> kQ := PathAlgebra(Rationals, Q);;
gap> M := RightModuleOverPathAlgebra
      (kQ, [0,1,1], [{"b", [[5]]}]);;
gap> N := RightModuleOverPathAlgebra
      (kQ, [1,2,1], [{"a", [[0,3]]},
                     [{"b", [[1],[1]]}]]);;
```

# Module homomorphisms in QPA

$$\begin{array}{ccccc}
 M: & 0 & \longrightarrow & k & \xrightarrow{5} & k \\
 & \downarrow & & \downarrow & & \downarrow \\
 & h & & (3 \ 2) & & (1) \\
 N: & k & \longrightarrow & k^2 & \longrightarrow & k \\
 & & & (0 \ 3) & & 1
 \end{array}$$

```
gap> h := RightModuleHomOverAlgebra
      (M, N, [ [[0]], [[3,2]], [[1]] ]);
<<[ 0, 1, 1 ]> ---> <[ 1, 2, 1 ]>>
```